Three dimensional loop quantum gravity: physical scalar product and spin foam models

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Abstract

In this paper, we address the problem of the dynamics in three dimensional loop quantum gravity with zero cosmological constant. We construct a rigorous definition of Rovelli's generalized projection operator from the kinematical Hilbert space—corresponding to the quantization of the infinite dimensional kinematical configuration space of the theory—to the physical Hilbert space. In particular, we provide the definition of the physical scalar product which can be represented in terms of a sum over (finite) spin-foam amplitudes. Therefore, we establish a clear-cut connection between the canonical quantization of three dimensional gravity and spin-foam models. We emphasize two main properties of the result: first that no cut-off in the kinematical degrees of freedom of the theory is introduced (in contrast to standard 'lattice' methods), and second that no ill-defined sum over spins ('bubble' divergences) are present in the spin foam representation.

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1. Introduction

The goal of the *spin foam* approach [1] is to construct a mathematical well defined notion of path integral for loop quantum gravity as a devise for computing the '*dynamics*' of the theory. By '*dynamics*' here we mean the characterization of the kernel of the quantum constraints of the theory given by the quantization of the classical constraints of general relativity expressed in connection variables, namely:

$$G_i = D_a E^a_i, \qquad V_a = E^b_i F^i_{ab} \quad and \qquad S = \epsilon^{ij}{}_k E^a_i E^b_j F^k_{ab} \, . \tag{1}$$

These constraints are respectively the Gauss constraints G_i, the vectorial constraints V_a and the scalar constraint S (with zero cosmological constant). Loop quantum gravity aims at characterizing the space of solutions of quantum Einstein's equations represented, in the canonical framework, by the previous set of constraints (for recent reviews see [2, 3, 4] and the beautifully new book by Rovelli [5]; for an extensive and deep description of the mathematical structure of the theory see the book of Thiemann [6]). Solutions of Gauss and vectorial constraints are well understood and are described in terms of spin-networks. They form a Hilbert space $\mathcal{H}_{kin}$ whose scalar product is defined by the Ashtekar-Lewandowski (A.L.) measure and is denoted $<,>$ in the following. The scalar constraint is much more involved to implement and is still problematic.

Spin foam models have been studied as an attempt to give an explicit construction of the generalized projection operator P from $\mathcal{H}_{kin}$ into the kernel of the quantum scalar constraint. The resolution of constraint systems using the generalized projection operator P (also called *rigging map*) has been studied in great generality by many authors. For references and a description of the main ideas and references see [7]. The technique was used to solve the diffeomorphism constraint in loop quantum gravity in [8]. The idea that the generalized projection operator P onto the kernel of the scalar constraint can be represented as a sum over spin foam amplitudes was introduced in [9, 10]. In this latter work the authors investigate a regularization of the formal projector into the kernel of the scalar constraint $S(x)$, given by

$$P = \text{``}\prod_{x \in \, \Sigma} \delta(\hat{\mathcal{S}}(x))\text{''} = \int D[N] \, \exp(i \int_{\Sigma} N\hat{\mathcal{S}}) \, , \tag{2}$$

was presented. One can also define the notion of path integral for gravity as a lattice discretization of the formal path integral for GR in first order variables

$$P = \int \; D[e] \; D[A] \; \mu[A,e] \, \exp\left[iS_{GR}(e,A)\right] \tag{3}$$

where the formal measure $\mu[A,e]$ must be determined by the Hamiltonian analysis of the theory. The issue of the measure [11] has been so far neglected in the definition of spin foam models of this type. On the other hand, no much progress in developing the formal definition of (2) proposed in [9, 10] has been attained in part because of

the ambiguities present in the construction of the scalar constraint, and also because of open questions regarding the finiteness of the proposed regularization. To see this in more detail let us briefly describe the main idea introduced by Rovelli and Reisenberger in the context of loop quantum gravity. In this case one is concerned with the definition of (2) in the canonical framework. Given two *spin network* states s, s' the physical scalar product $< s, s' >_{ph} := < Ps, s' >$ can be formally defined by

$$\langle Ps, s' \rangle = \int D[N] \sum_{n=0}^{\infty} \frac{i^n}{n!} < \left[\int_{\Sigma} N(x) \hat{\mathcal{S}}(x) \right]^n s, s' >, \tag{4}$$

where the exponential in (2) has been expanded in powers. From early on, it was realized that smooth loop states are naturally annihilated (independently of any regularization ambiguity) by $\hat{\mathcal{S}}$ [12, 13] [1]. In fact, one can show that $\hat{S}$ acts only on *spin network* nodes. Generically, it does so by creating new links and nodes modifying in this way the underlying graph of the *spin network* states [14, 15]. The action of $\hat{\mathcal{S}}$ can be visualized as an 'interaction vertex' in the time evolution of the node (Figure 5). Therefore, each term in the sum (4) represents a series of transitions–given by the local action of $\hat{\mathcal{S}}$ at *spin network* nodes–through different *spin network* states interpolating the boundary states s and s' respectively. They can in fact be expressed as sum over 'histories' of *spin networks* that can be pictured as a system of branching surfaces described by a 2-complex. In this picture, links 'evolve' to form 2-dimensional faces (that inherit the corresponding spin label) and nodes evolve into 1-dimensional edges (inheriting intertwiners). The action of $\hat{\mathcal{S}}$ generates vertices were new nodes are created and a transition to a different *spin network* takes place. Every such history is a *spin foam* (see Figure 5).

Before even considering the issue of convergence of this series, the problem with this definition is evident: every single term in the sum is a divergent integral! Therefore, this way of presenting *spin foams* has to be considered as formal until a well defined regularization of (2) is provided. Possible regularization schemes are discussed in [10] although they have not been implemented in concrete examples.

The underlying discreteness discovered in loop quantum gravity is crucial: in *spin foam* models the functional integral for gravity is replaced by a sum over amplitudes of combinatorial objects given by foam-like configurations (*spin foams*). The precise definition was first introduced by Baez in [16]. A *spin foam* represents a possible history of the gravitational field and can be interpreted as a set of transitions through different quantum states of space. Boundary data in the path integral are given by polymer-like excitations (*spin network* states) representing 3-geometry states in loop quantum gravity.

Pure gravity in three dimensions is a well studied example of integrable system that can be rigorously quantized (for a review see [17]). The reason for that is the fact that GR in three dimensions does not have local degrees of freedom. The degrees

[1] This set of states is clearly too small to represent the physical Hilbert space (e.g., they span a zero volume sector).

of freedom are topological and therefore finitely many. Different quantization schemes have been explored and one can say that we have a well understanding of three dimensional quantum gravity. From our perspective three dimensional gravity is taken as an toy model for the application of quantization techniques that are expected to be applicable in four dimensions (see [18] for a novel proposal and an account and a discussion technical difficulties arising in four dimensions). In this sense we want to quantize the theory according to Dirac prescription which implies having to deal with the infinitely many degrees of freedom of a field theory at the kinematical level, i.e., we want to quantize first and then reduce at the quantum level. This is precisely the avenue that is explored by loop quantum gravity in four dimensions.

The next section is devoted to a brief review of the canonical formulation of three dimensional gravity à la loop gravity. However, our approach is a bit different than the usual loop gravity description for we treat the vectorial constraints and the scalar constraint together in the curvature constraint $F_{ab} = 0$. The Gauss constraint still exists and the kinematical Hilbert space $\mathcal{H}_{kin}$ is defined by the set of its solutions endowed with the A.L. measure. In section 3, we address the problem of the dynamics, i.e. we characterize the physical Hilbert space $\mathcal{H}_{phys}$ by providing a regularization of the Rovelli's generalized projection operator. The physical scalar product can be represented a a sum over spin foams whose amplitudes coincide—when restricted in a suitable way— with the definition of the covariant path integral on a triangulation: the Ponzano-Regge model. Then, we propose a basis for the physical Hilbert space. Finally, we end up with a discussion on the possible generalization of these results (the most natural case being the case of a non-vanishing cosmological constant) and on the possibility to adapt this method to the four dimensional framework.

2. Canonical three dimensional gravity

The theory we are interested in is three dimensional gravity in first order formalism. The space-time $\mathcal{M}$ is a three dimensional oriented smooth manifold and the action is simply given by

$$S[e,\omega] = \int_{\mathcal{M}} \mathrm{Tr}[e \wedge F(\omega)] \tag{5}$$

where e is the triad, i.e. a Lie algebra ($\mathfrak{g}$) valued 1-form, $F(\omega)$ is the curvature of the three dimensional connection ω and Tr denotes a Killing form on $\mathfrak{g}$. For simplicity we will concentrate on Riemannian gravity so the previous fields should be thought as defined on $SU(2)$ principal bundle over $\mathcal{M}$ and the Lie algebra is $\mathfrak{g} = su(2)$. We assume the space time topology to be $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ is a Riemann surface of arbitrary genus.

2.1. Phase space, constraints and gauge symmetries

Upon the standard 2+1 decomposition, the phase space in these variables is parametrized by the pull back to Σ of ω and e. In local coordinates we can express them in terms of the 2-dimensional connection A_a^i and the triad field $E_j^b = \epsilon^{bc} e_c^k \eta_{jk}$ where $a = 1, 2$ are space coordinate indices and $i, j = 1, 2, 3$ are $su(2)$ indices. The symplectic structure is defined by

$$\{A_a^i(x), E_j^b(y)\} = \delta_a^{\,b}\ \delta_{\,j}^i\ \delta^{(2)}(x, y).$$

Local symmetries of the theory are generated by the first class constraints $D_b E_j^b \simeq 0$ and $F_{ab}^i(A) \simeq 0$. More precisely, if we smear them out with arbitrary test fields α and λ, the constraints read:

$$G[\alpha, A, E] = \int\limits_{\Sigma} \alpha^j D_b E_j^b = 0 \quad \text{and} \quad C[\lambda, A] = \int\limits_{\Sigma} \lambda_j F_{ab}(A)^j = 0. \tag{6}$$

In the sequel, we will assume that the test fields α and λ do not depend on the phase space variables. The Gauss constraint $G[\alpha, A, E]$ generates infinitesimal $SU(2)$ gauge transformations

$$\delta_\alpha A_a^i = \{A_i^a, G[\alpha, A, E]\} = (D_a \alpha)^i, \quad \delta_\alpha E_i^a = \{E_i^a, G[\alpha, A, E]\} = \alpha^k E^{ja} \epsilon_{ijk}, \tag{7}$$

while the curvature constraint $C[\lambda, A]$ generates the following transformations:

$$\delta_\lambda A_a^i = \{A_a^i, C[\lambda, A]\} = 0, \quad \delta_\lambda E_i^a = \{E_i^a, C[\lambda, A]\} = \epsilon^{ac} D_c \lambda \tag{8}$$

As it is well known, diffeomorphisms are contained in the previous transformations. More precisely, given a vector field $v = v^a \partial_a$ on the surface Σ, one defines the parameters $\alpha^i(v) = v^a A_a^i$ and $\lambda_i(v) = \epsilon_{ab} E_i^a v^b$ and we have

$$(\mathcal{L}_v A)_a^i \simeq \delta_{\alpha(v)} A_i^a, \quad (\mathcal{L}_v E)_i^a \simeq \delta_{\alpha(v)} E_a^i + \delta_{\lambda(v)} E_a^i \tag{9}$$

where $\mathcal{L}_v$ is the Lie derivative operator along the vector field v and $\simeq$ denotes weak equalities, i.e., valid on the constraint surface. Note that gauge transformations are equivalent to diffeomorphisms if and only if we restrict the E-field to be non-degenerate.

2.2. Kinematical Hilbert space

In analogy with the four dimensional case we follow Dirac's procedure and in order to quantize the theory we first find a representation of the basic variables in an auxiliary Hilbert space $\mathcal{H}_{aux}$. The basic functionals of the connection are represented by the set of holonomies along paths $\gamma \subset \Sigma$. Given a connection A and a path γ, one defines the holonomy $h_\gamma[A]$ by

$$h_\gamma[A] = P \exp \int_\gamma A\ . \tag{10}$$

As for the triad, its associated basic variable is given by the smearing of E along co-dimension 1 surfaces. One promotes these basic variables to operators acting on an auxiliary Hilbert space where constraints are represented. The physical Hilbert space is defined by those 'states' that are annihilated by the constraints. As these 'states' are not normalizable with respect to the auxiliary inner product they are not in $\mathcal{H}_{aux}$ and have to be regarded rather as distributional.

The auxiliary Hilbert space is defined by the Cauchy completion of the space of cylindrical functionals Cyl, on the space of (generalized) connections $\bar{\mathcal{A}}$[2]. The space Cyl is defined as follows: any element of Cyl, $\Psi_{\Gamma,f}[A]$ is a functional of A labeled by a finite graph $\Gamma \subset \Sigma$ and a continuous function $f : SU(2)^{N_\ell(\Gamma)} \to \mathbb{C}$ where $N_\ell(\Gamma)$ is the number of links of the graph Γ. Such a functional is defined as follows

$$\Psi_{\Gamma,f}[A] = f(h_{\gamma_1}[A], \cdots, h_{\gamma_{N_\ell(\Gamma)}}[A]) \tag{11}$$

where $h_{\gamma_i}[A]$ is the holonomy along the link γ_i of the graph Γ. If one considers a new graph Γ' such that $\Gamma \subset \Gamma'$, then any cylindrical function $\Psi_{\Gamma,f}[A]$ trivially corresponds to a cylindrical function $\Psi_{\Gamma',f'}[A]$ [19].

For example, let us consider the path $\alpha(t)$ given by $\alpha : [0,3] \to \Sigma$. We define Γ to be the graph given by the single link $\gamma = \{\alpha(t)\ for\ t \in [0,2]\}$ and by the two nodes $\alpha(0)$ and $\alpha(2)$. The graph Γ' is defined by the three links $\gamma_i = \{\alpha(t)\ for\ t \in [i-1,i]\}$ and the four nodes $\alpha(0), \alpha(1), \alpha(2)$ and $\alpha(3)$. Clearly $\Gamma \subset \Gamma'$. Therefore, given $\Psi_{\Gamma,f}[A] = f(h_\gamma[A])$, we can construct $\Psi_{\Gamma',f'}[A] = f'(h_{\gamma_1}[A], h_{\gamma_2}[A], h_{\gamma_3}[A])$ such that $\Psi_{\Gamma,f}[A] = \Psi_{\Gamma',f'}[A]$ by choosing $f'(h_{\gamma_1}[A], h_{\gamma_2}[A], h_{\gamma_3}[A]) = f(h_{\gamma_1}[A]h_{\gamma_2}[A])$.

In general, for any two cylindrical functions $\Psi_{\Gamma_1,f}[A]$ and $\Psi_{\Gamma_2,g}[A]$, the inner product is defined by the Ashtekar-Lewandowski measure

$$\begin{aligned} &\mu_{AL}(\overline{\Psi_{\Gamma_1,f}[A]}\Psi_{\Gamma_2,g}[A]) =< \Psi_{\Gamma_1,f}, \Psi_{\Gamma_2,g} > \\ &:= \int \prod_{i=1}^{N_{\ell_{\Gamma_{12}}}} dh_i \overline{f(h_{\gamma_1}, \cdots, h_{\gamma_{N_\ell(\Gamma_{12})}})} g(h_{\gamma_1}, \cdots, h_{\gamma_{N_\ell(\Gamma_{12})}}) \end{aligned} \tag{12}$$

where dh_i corresponds to the invariant $SU(2)$-Haar measure, $\Gamma_{12} \subset \Sigma$ is a graph containing both Γ_1 and Γ_2, and we have used the same notation f (resp. g) to denote the extension of the function f (resp. g) on the graph Γ_{12}. The auxiliary Hilbert space $\mathcal{H}_{aux}$ is defined as the Cauchy completion of Cyl under (12).

The (generalized) connection is quantized by promoting the holonomy (10) to an operator acting by multiplication in $\mathcal{H}_{aux}$ as follows:

$$\widehat{h_\gamma[A]}\Psi[A] \ = \ h_\gamma[A]\Psi[A]\ . \tag{13}$$

It is easy to check that the quantum holonomy is a self adjoint operator in $\mathcal{H}_{aux}$. The triad is promoted to a self adjoint operator valued distribution that acts as a

[2]A generalized connection is a map from the set of paths $\gamma \subset \Sigma$ to $SU(2)$. It corresponds to an extension of the notion of holonomy $h_\gamma[A]$ introduced above.

derivation, namely:

$$\hat{E}^j_a = -i\ell_p \epsilon_{ab}\eta^{kj}\frac{\delta}{\delta A^k_b} \quad , \tag{14}$$

where $\ell_p = \hbar G$ is the Planck length in three dimensions. In terms of the triad operator we can construct geometric operators corresponding to the area of regions in Σ or the length of curves [20, 21, 23]. So far we have not specified the space of graphs that we are considering.

The Gauss constraint (6) can be defined in terms of the basic variables introduced above. It generates gauge transformations whose action on Cyl transforms the holonomy as follows

$$h_\gamma[A] \longmapsto g_s h_\gamma[A] g_t^{-1} \tag{15}$$

where $g_s, g_t \in SU(2)$ are group elements associated to the *source* and *target* nodes of γ respectively. The so-called *kinematical* Hilbert space $\mathcal{H}_{kin} \subset \mathcal{H}_{aux}$ is defined by the states in $\mathcal{H}_{aux}$ which are gauge invariant and hence in the kernel of the Gauss constraint.

One can introduce an orthonormal basis of states in $\mathcal{H}_{aux}$ using $SU(2)$ harmonic analysis. Namely, any (Haar measure) square integrable function $f : SU(2) \to \mathbb{C}$ can be expanded in terms of unitary irreducible representations of $SU(2)$

$$f(h) = \sum_j f_j \overset{j}{\Pi}(h) \; , \tag{16}$$

where the Fourier component $f_j = \int dh\, \overline{\overset{j}{\Pi}(h)} f(h)$ can be viewed as an element of $j^* \otimes j$ (we use the same notation for the representation and its associated vector space). The straightforward generalization of this decomposition to functions $f : SU(2)^N \to \mathbb{C}$ allows us to write any cylindrical function (11) as

$$\Psi_{\Gamma,f}[A] = \sum_{j_1\cdots j_{N_\ell}} f_{j_1\cdots j_{N_\ell}} \overset{j_1}{\Pi}(h_{\gamma_1}[A]) \cdots \overset{j_{N_\ell}}{\Pi}(h_{\gamma_{N_\ell(\Gamma)}}[A]), \tag{17}$$

where Fourier components $f_{j_1\cdots j_{N_\ell(\Gamma)}}$ are elements of $(\otimes_{i=1}^{N_\ell(\Gamma)} j_i) \otimes (\otimes_{i=1}^{N_\ell(\Gamma)} j_i)^*$.

We can write any element of Cyl as a linear combination of the tensor product of $N_\ell(\Gamma)$ $SU(2)$ irreducible representations. Using the definition of the scalar product and (16) one can easily check that these elementary cylindrical functionals are orthogonal.

Gauge transformations (15) generated by the Gauss constraint induce a gauge action on Fourier modes which simply reads:

$$\overset{j}{\Pi}(h) \rightarrow \overset{j}{\Pi}(g_s) \, \overset{j}{\Pi}(h) \, \overset{j}{\Pi}(g_t^{-1}).$$

A basis of gauge invariant functions can then be constructed by contracting the tensor product of representation matrices in Equation (17) with a $su(2)$-invariant

tensors or $su(2)$-intertwiners. If we select an orthonormal basis of intertwiners $\iota_n \in \mathrm{Inv}[j_1 \otimes j_2 \otimes \cdots \otimes j_{N_\ell}]$ where n labels the elements of the basis we can write a basis of gauge invariant elements of Cyl called the *spin network basis*. Each spin network is labeled by a graph $\Gamma \subset \Sigma$, a set of spins j_γ labeling links γ of the graph Γ and a set of intertwiners ι_n labeling nodes n of the graph Γ, namely:

$$s_{\Gamma,\{j_\ell\},\{\iota_n\}}[A] = \bigotimes_{n\in\Gamma} \iota_n \bigotimes_{\gamma\in\Gamma} \overset{j_\gamma}{\Pi}\left(h_\gamma[A]\right). \tag{18}$$

We have represented a spin network state in Figure (1). In order to lighten notations, we will omit indices (the graph, representations and intertwiners) labeling spin-networks in the sequel.

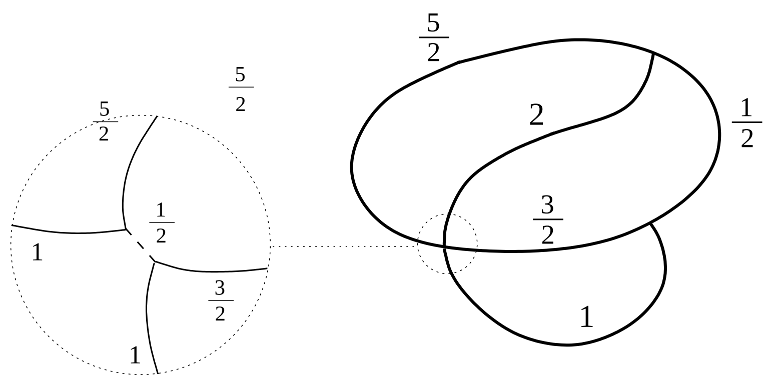


Figure 1: Illustration of a spin-network state. At 3-valent nodes the intertwiner is uniquely specified by the corresponding spins. At 4 or higher valent nodes an intertwiner has to be specified. Choosing an intertwiner corresponds to decompose the n-valent node in terms of 3-valent ones adding new virtual links (dashed lines) and their corresponding spins. This is illustrated explicitly in the figure for the 4-valent node.

3. Dynamics and spin foams

In this section we completely solve $2+1$ Riemannian quantum gravity by providing a regularization of the Rovelli's generalized projection operator P. This allows us to construct the physical scalar product: we present its spin foam representation and its relationship with the Ponzano-Regge model. Finally, we provide a complete characterization of the physical Hilbert space $\mathcal{H}_{phys}$ which is analogous to the one provided in [24]. Moreover, the spin foam representation allow us to find a basis in $\mathcal{H}_{phys}$.

3.1. Regularization of P

In this section we introduce the regularization of the generalized projection operator P and show how the matrix elements $< Ps, s' >$ admit a spin foam state

sum representation that is independent of the regularization and it is not based on any auxiliary structure such as a triangulation or cellular decomposition when the regulator is removed.

We start with the formal expression

$$P = \text{``}\prod_{x\in\ \Sigma}\delta(\hat{F}(A))\text{''} = \int D[N]\ \exp(i\int_{\Sigma}\mathrm{Tr}[N\hat{F}(A)])\ , \tag{19}$$

where $N \in su(2)$. We now introduce a regularization of (19). We will give a definition of P by providing a regularization of its matrix elements $< Ps, s' >$ for any pair of spin network states $s, s' \in \mathcal{H}_{kin}$. Let's denote by Γ and Γ' the graphs on which s and s' are defined respectively. We introduce an arbitrary cellular decomposition of Σ denoted $\Sigma_{\epsilon}^{\Gamma\Gamma'}$, where $\epsilon \in \mathbb{R}$, such that:

1. The graphs Γ and Γ' are both contained in the graph defined by the union of 0-cells and 1-cells in $\Sigma_{\epsilon}^{\Gamma\Gamma'}$.

2. For each individual 2-cells (plaquette) p there exist a ball $\mathcal{B}_{\epsilon}$ of radius ϵ—defined using the local topology—such that $p \subset \mathcal{B}_{\epsilon}$.

Consequently all 2-cells shrink to zero when $\epsilon \to 0$.

Based on the cellular decomposition $\Sigma_{\epsilon}^{\Gamma\Gamma'}$ we can now define $< Ps, s' >$ by introducing a regularization of the right hand side of (19). Given two spin networks s and s' based on the graphs Γ and Γ', we order the set of plaquettes $p^i \in \Sigma_{\epsilon}^{\Gamma\Gamma'}$ for $i = 1, \cdots N_p^{\epsilon}$ where N_p^{ϵ} is the total number of plaquettes for a given ϵ. We define the physical scalar product between s and s' as

$$< s, s' >_p = < Ps, s' > := \lim_{\epsilon\to 0}\ \sum_{j_{p^i}}(2j_{p^i}+1)\ < \prod_{p^i}\chi_{j_{p^i}}(U_{p^i})\ s, s' >, \tag{20}$$

where the sum is over all half-integers j_{p^i} labelling each plaquette, U_{p^i} is the holonomy around p^i (based on an arbitrary starting point) and $\chi_{j_{p^i}}(U_{p^i})$ is the trace in the j_{p^i} representation.

Notice that each term in the previous sum is the matrix element of a self adjoint operator in $\mathcal{H}_{kin}$ and therefore well defined. The question is whether the previous expression is well defined. As we will see in the sequel (see remarks at the end of this section), for a fixed value of ϵ, the sum inside the limit is convergent for any Riemann surface of genus $g \geq 2$. We also show in Section 3.2.1. that the result is independent of the regulator ϵ and therefore the limit exists trivially. In the case of the sphere $g = 0$ and the torus $g = 1$ the previous sum is divergent; we discuss the regularization of the physical inner product in these special cases in Section **??**.

Let us now give the motivation for the definition of the inner product above. We consider a local patch $U \subset \Sigma$ where we choose the cellular decomposition to be

square with cells of coordinate length ϵ. In that patch, the integral in the exponential in (19) can be written as a Riemann sum

$$F[N] = \int\limits_U \mathrm{Tr}[NF(A)] = \lim_{\epsilon \to 0} \sum_{p^i} \epsilon^2 \mathrm{Tr}[N_{p^i} F_{p^i}],$$

where p^i labels the i^{th} plaquette and $N_{p^i} \in su(2)$ and $F_{p^i} \in su(2)$ are values of $N^j \tau_j$ and $\tau_j \epsilon^{ab} F^j_{ab}[A]$ at some interior point of the plaquette p^i and ϵ^{ab} is the Levi-Civita tensor. The basic observation is that the holonomy $U_{p^i} \in SU(2)$ around the plaquette p^i can be written as

$$U_{p^i}[A] = \mathbb{1} + \epsilon^2 F_{p^i}(A) + \mathcal{O}(\epsilon^2)$$

which implies

$$F[N] = \int\limits_U \mathrm{Tr}[NF(A)] = \lim_{\epsilon \to 0} \sum_{p^i} \mathrm{Tr}[N_{p^i} U_{p^i}[A]] \ , \tag{21}$$

where the Tr in the r.h.s. is taken in the fundamental representation. Notice that the explicit dependence on the regulator ϵ has dropped out of the sum on the r.h.s., a sign that we should be able to remove the regulator upon quantization. With all this it is natural to write the following formal expression for the generalized projection operator:

$$\begin{aligned} < Ps, s' > \ &= \ \lim_{\epsilon \to 0} \ < \prod_{p^i} \int \ dN_{p^i} \ \exp(i \mathrm{Tr}[N_{p^i} \hat{U}_{p^i}]) s, \ s' > \\ &= \ \lim_{\epsilon \to 0} \ < \prod_{p^i} \ \delta(U_{p^i}) s, \ s' >, \end{aligned} \tag{22}$$

where the last equality follows from direct integration over N_{p^i} at the classical level; $\delta(U)$ being the distribution such that $\int dg \ f(g) \delta(g) = f(\mathbb{1})$ for $f \in \mathcal{L}^2(SU(2))$. We can write $\delta(U)$ as a sum over unitary irreducible representations of $SU(2)$, namely

$$\delta(U_{p^i}) = \sum_j \Delta_j \ \chi_j(U_{p^i}), \tag{23}$$

where $\Delta_j = 2j + 1$ is the dimension of the j-representation and, at the classical level, $\chi_j(U)$ is the character or trace of the j-representation matrix of $U \in SU(2)$. In contrast to the formal example in four dimensions, mentioned in the introduction, the previous expansion has more chances to have a precise meaning in the quantum theory as each term in the sum can be promoted to a well defined self-adjoint operator in $\mathcal{H}_{kin}$: it corresponds to the Wilson loop operator in the j-representation around plaquettes corresponding to the quantization introduced in (13).

The object $\delta(U_{p^i})$ is clearly not well defined as an operator in $\mathcal{H}_{kin}$ as for any $\psi \in \mathcal{H}_{kin}$ the state $\delta(U_{p^i})\psi$ is non-normalizable. This fact might seem as a problem for the definition of the generalized projection operator P. This is however a standard feature which appears in dealing with group averaging techniques when the constraints generate non compact orbits (as clearly is the case with the curvature constraint). Solutions of the curvature constraint lie outside $\mathcal{H}_{kin}$. Thus $\mathcal{H}_{phys}$ is not a subspace of $\mathcal{H}_{kin}$. Physical states correspond to 'distributional states' in Cyl^*, the dual of the dense set $Cyl \subset \mathcal{H}_{kin}$. Through the inner product we can identify elements $|s>$ of a spin network basis in Cyl to dual spin networks $<s| \in Cyl^*$. The generalized projector operator $P: Cyl \to Cyl^*$ takes any element $s \in Cyl$ and sends it to a linear form $Ps \in Cyl^*$; we denote the action of this linear form on any element $s' \in Cyl$ by $Ps(s') :=< Ps, s' >$. The physical inner product is defined as $< a, b >_p :=< Pa, b >$. As a result $\mathcal{H}_{phys}$ can be viewed as the set of equivalence classes elements of Cyl^* of the form $< Ps|$ where $< Ps| \sim < Ps'|$ if they are equal as linear forms. The fact that the limit (20) exists for $g \geq 2$ for any $s, s' \in Cyl$ shows that the map P is well defined.

Instead of working all the time with the explicit expression for the physical inner product (20) it will be often convenient to use the more compact expression of P in terms of product of delta distributions as in (22).

We conclude this section with some remarks:

Remark 1: The argument of the limit in (20) satisfies the following bound

$$| < s, s' >_p | = \left| \sum_{j_{p^i}} (2j_{p^i} + 1) \ \mu_{AL} \left(\prod_{p^i} \chi_{j_{p^i}}(U_{p^i}[A]) \ \overline{s[A]} s'[A] \right) \right| \leq C \sum_j (2j+1)^{2-2g}, \tag{24}$$

where C is a real positive constant.[3] The convergence of the sum for $g \geq 2$ follows directly.

Remark 2: The case of the sphere $g = 0$ is easy to regularize. In this case (20)

[3]Let us define a regularization of the $SU(2)$ delta distribution as follows. Take a one parameter family of (suficiently smooth) functions $\delta_\Delta(g)$ such that $\delta_\Delta(g) \geq 0$ for all $g \in SU(2)$ and $\Delta \in \mathbb{R}^+$ and $\delta(g) = \lim_{\Delta \to 0} \delta_\Delta(g)$ in the sense of distributions. We have the following identity

$$< \prod_p \sum_{j_p} (2j_p + 1) \chi_{j_p}(U_p) s, s' > \ = \lim_{\Delta \to 0} < \prod_p \delta_\Delta(U_p) s, s' > , \tag{25}$$

where we have used two equivalent definitions of the delta function. Due to the boundeness of the elements of Cyl we have

$$R_1 \leq \mathrm{Re} \left[\overline{s[A]} s'[A] \right] \leq R_2 \quad \text{and} \quad I_1 \leq \mathrm{Im} \left[\overline{s[A]} s'[A] \right] \leq I_2$$

where $R_1, R_2, I_1, I_2 \in \mathbb{R}$. Using the positivity of the regularization of the delta distribution we can write

diverges simply because of a redundancy in the product of delta distributions in the notation of (22). This is a consequence of the discrete analog of the Bianchi identity. It is easy to check that eliminating a single arbitrary plaquette holonomy from the product in (20) makes P well defined and produces the correct (one dimensional) $\mathcal{H}_{phys}$.

The case of the torus $g = 1$ is more subtle; we will discuss it at the end of the paper.

Remark 3: Clearly the order of the product of δ-distributions is not important, as the operators being multiplied are all commuting. For this reason we will consider a symmetrized version of the previous regularization that it is trivially equivalent to (22) in this case [4]. Thus from now on we will consider the following expression of P :

$$< Ps, s' >:= \lim_{\epsilon \to 0} \frac{1}{N_p^{\epsilon}!} \sum_{\sigma} \ \ < \prod_{p^{\sigma(i)}} \ \delta(U_{p^{\sigma(i)}})s, \ s' >, \tag{26}$$

where N_p^{ϵ} is the ϵ-dependent number of plaquettes and $\sigma(\{i\})$ denotes a permutation of the set of plaquette labels. Now we introduce some notation. We denote by $< P_{\epsilon}s, s' >$ the argument of the limit above which we can think of as a truncated version of P, namely:

$$< P_{\epsilon}\hat{s}, s' > := \frac{1}{N_p^{\epsilon}!} \sum_{\sigma(\{i\})} \ \ < \prod_{p^{\sigma(i)}} \ \delta(U_{p^{\sigma(i)}})s \ s' > . \tag{27}$$

We denote by $< P_{\epsilon}^{\sigma}s, s' >$ each of the possible orderings σ of delta distributions in the previous expression, explicitly:

$$< P_{\epsilon}^{\sigma}\hat{s}, s' > :=< \prod_{p^{\sigma(i)}} \ \delta(U_{p^{\sigma(i)}})s \ s' > . \tag{28}$$

$$\begin{aligned} R_1 \sum_j (2j+1)^{2-2g} &= R_1 \lim_{\Delta \to 0} < \prod_p \delta_{\Delta}(U_p); 1 >\leq \\ &\leq \lim_{\Delta \to 0} \mathrm{Re} < \prod_p \delta_{\Delta}(U_p)s; s' >\leq \\ &\leq R_2 \lim_{\Delta \to 0} < \prod_p \delta_{\Delta}(U_p); 1 >= R_2 \sum_j (2j+1)^{2-2g}, \end{aligned}$$

where the evaluation $\lim_{\Delta \to 0} < \prod_p \delta_{\Delta}(U_p); 1 >= \sum_j (2j+1)^{2-2g}$ follows directly from the definition of the AL-measure, see [22]. A similar bound holds for the imaginary part from where one can deduce (24) using the fact that

$$\sum_{j_p} (2j_p+1) < \prod_p \chi_{j_p}(U_p)s, s' >=< \prod_{j_p} \sum_{j_p} (2j_p+1)\chi_{j_p}(U_p)s, s' >,$$

once the convergennce of the r.h.s. is granted.

[4] As we shall shortly see, each of the δ distribution produces a local transition in the spin foam representation. Different orderings correspond to different 'coordinate time' evolutions and produce different consistent (histories) *spin foams*. Now all these spin foams have the same amplitude in this theory but in a general setting should represent independent contributions to the path integral.

Remark 4: It is immediate to see that (20) satisfies hermitian condition

$$< Ps, s' >= \overline{< Ps', s >}. \tag{29}$$

Remark 5: The positivity condition also follows from the definition $< Ps, s > \geq 0$.

3.2. Spin foam representation

As in the case of the standard Feynman derivation of the path integral representation of the evolution operator, the spin foam representation of the generalized projection operator follows from inserting resolutions of the identity in $\mathcal{H}_{kin}$ given by

$$\mathbb{1} = \sum_{\Gamma \subset \Sigma, \{j\}_\Gamma} |\Gamma, \{j\} >< \Gamma, \{j\}| \tag{30}$$

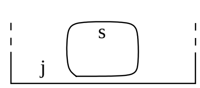
between consecutive delta distributions in (26). In the previous expression the sum is taken over all the elements of a chosen spin network basis in $\mathcal{H}_{kin}$. Because of the definition of the Ashtekar-Lewandowski measure only those spin networks that differ by a single plaquette loop contribute in following steps making the transition amplitudes finite.

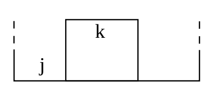
In order to visualize in which way the spin foam representation arises, let us consider a simple example. For concreteness we assume that the value of the regulator ϵ is fixed at some non zero value. Consider the regularized matrix element $< P^{\sigma}_{\epsilon} s, s' >$ between the spin networks s and s' based on the loop $\ell \in \Sigma$ and labeled by the representations j and m respectively (see Figure 2). At this finite regulator ϵ we have 9 plaquettes covering the interior of the loop. Taking the definition of $< P^{\sigma}_{\epsilon} s, s' >$ and inserting resolutions of identities between the plaquette delta distributions it is easy to show that from the large set of graphs in Σ only a finite number of intermediate graphs survive in the computation of $< P^{\sigma}_{\epsilon} s, s' >$.

Let us study the first step in the sequence of transitions illustrated in Figure 2. We consider the ordering of plaquettes such that the first delta distribution in the regularization of P (one term in the sum over orderings of equation (26)) is evaluated on the lower central plaquette with respect to the loop ℓ. The first delta function acts on the loop ℓ as

$\delta(U_p)$ 

$=$

$\sum_s \Delta_s$ 

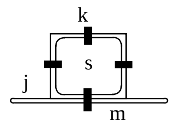
$= \sum_{k,s} \Delta_k N_{j,m,k} \delta_{k,s}$ 

(31)

where on the right hand side we have expanded the result of the action of the Wilson loop operator in the representation s on our initial state in terms of spin network

states. The coefficient of the expansion is determined by the AL measure, namely

$$N_{j,m,k}\delta_{k,s} = \quad (32)$$



,

where we are using the graphical notation described in the appendix (the Haar measure integration is denoted by a dark box overlapping the different representation lines (see (42))). It is a simple exercise to evaluate the previous diagram resulting in $N_{j,m,k} = 1$ if j, k, m are compatible spins and $N_{j,m,k} = 0$ otherwise. Equation (31) implies that when inserting the resolution of the identity (30) between the first and second delta distributions the corresponding term in $< P^{\sigma}_{\epsilon} s, s' >$ only the intermediate states on the right hand side of (31) will survive. The first delta distribution acts on the initial state by attaching a new infinitesimal loop. The following deltas would have a similar effect creating new infinitesimal loops associated to each corresponding plaquettes. In this way each term in $< P_{\epsilon} s, s' >$ 'explores' the set of intermediate spin network states based only on those graphs that are contained in the regulating cellular decomposition $\Sigma^{\Gamma\Gamma'}_{\epsilon}$. In this set of transitions $SU(2)$ gauge invariance is preserved which is manifested as spin compatibility conditions. Finally, when we contract with the final state s' only the sequence of spin network states which represent consistent histories remain. The consistency condition is precisely given by the requirement that the set of transitions produces a spin foam [25].

An important fact is that these discrete spin foams do not contain bubbles. Once a new loop labeled by a non-trivial spin is created it cannot be annihilated. The Wilson loops operators in the series that define the delta distributions are self adjoint operators and in this way they can both create or annihilate infinitesimal loops. However, they can act only in one direction when sandwiched between fixed boundary states. In order to produce a spin foam with a bubble we would need to act twice with the same Wilson loop which amounts to acting twice with the delta distribution; a clearly ill defined operation. In fact it would lead to a divergence proportional to $\delta(\mathbb{1}) = \sum_s (2s+1)^2$. This is precisely the kind of bubble divergences obtained in the Ponzano-Regge model! No such divergences are present in our definition of P as each plaquette delta distribution acts only once (26).

Therefore, the AL measure selects only those intermediate spin networks which provide a sequence of transitions between the 'initial' and 'final' state that can be represented by a spin foam with no bubble. In the case at hand and considering a particular arbitrary ordering of the plaquette delta distributions one of such sequence is illustrated in Figure 2. There is one such sequence for each value of k satisfying the spin compatibility conditions with j and m and its amplitude is given by Δ_k as we will explicitly shown. Notice that this is precisely the value of such spin foam in the Ponzano-Regge model. In the limit $\epsilon \to 0$ the discrete sequences contain more and more intermediate states and tend to a continuous history or continuous spin foam representation. One of these possible histories is depicted in Figure 3.

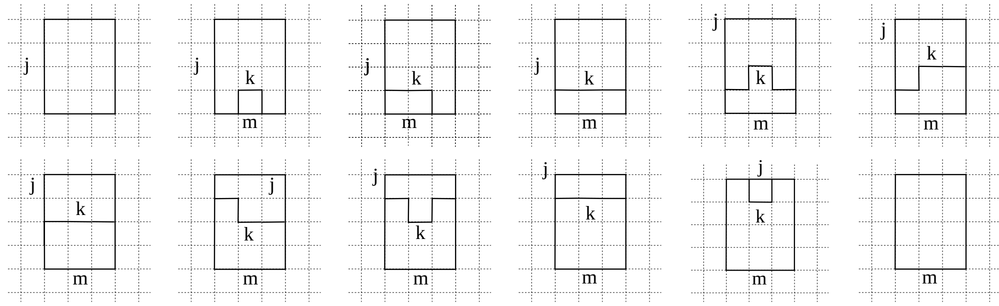


Figure 2: A set of discrete transitions representing one of the contributing histories implied by our regularization of the generalized projection P in Equation (26); from left to right in two rows.

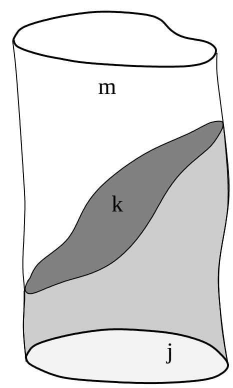


Figure 3: Spin foam representation of the transition between to loop states. Our regularization of the generalized projection operator P produces (in the path integral representation) a continuous transition between embedded spin networks. Here we illustrate the result at three different slicing.

Another example representing four valent vertex transition is represented in Figure 4. The continuous spin foam picture illustrated in Figure 5 is obtained when the regulator is removed in the limit $\epsilon \to 0$.

We will show in the sequel that in the limit when the regulator is removed we recover a spin foam representation of the matrix elements of the projection operator and hence the physical Hilbert space $\mathcal{H}_{phys}$ that is independent of any auxiliary structure.

3.2. Physical Hilbert space

In the previous sections we have shown how to obtain the spin foam representation of the physical inner product. We have also shown that our regularization of the generalized projector P implies the Ponzano-Regge amplitudes automatically in its gauge fixed form in the language of reference [26]. In this section we sum up the contributions of the infinite-many spin foams and provide a close formula for the

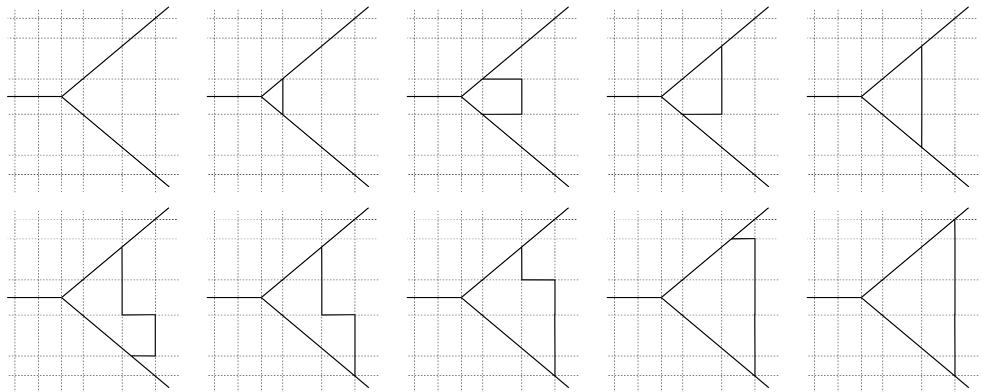


Figure 4: A set of discrete transitions representing one of the contributing histories implied by our regularization of the generalized projection P in Equation (26); from left to right in two rows.

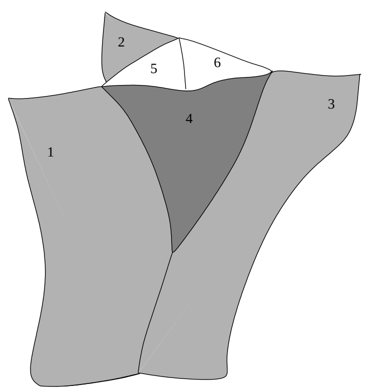


Figure 5: The regularization of the generalized projector P produces a continuous sequence of transitions through spin-network states that can be pictured in the form of a continuous 2-complex.

physical inner product for an arbitrary manifold of topology $\mathcal{M} = \Sigma \times \mathbb{R}$ where Σ is any Riemann surface. We also provide a basis of the physical Hilbert space.

3.2.1. Physical scalar product

In the computation of $< Ps, s' >$ the product of delta functions evaluated on the holonomy corresponding to infinitesimal plaquettes that do not intersect the graphs corresponding to s and s' can be integrated over to a single delta distribution that involves the holonomy around an irreducible loop. This is illustrated in Figure 6 where by the integration over the generalized connection on the common boundary of two adjacent plaquettes we can fusion the action of the corresponding two delta distributions into a single delta function associated to the union of the two plaquettes. We can continue this process in the computation of $< Ps, s' >$ until we reach

the plaquettes that intersect the spin network states s and s'

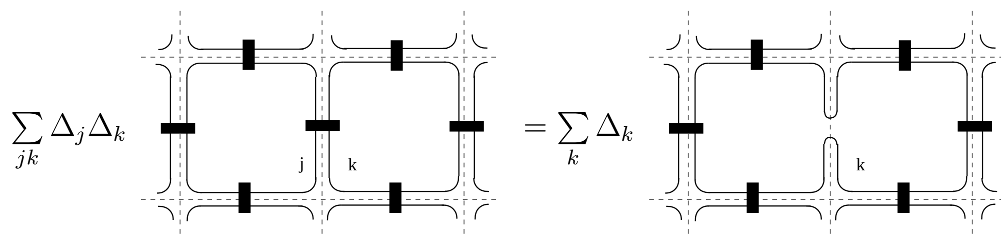


Figure 6: Infinitesimal plaquette-delta-distributions can be integrated over to the action of a single irreducible loop (see equation (44) in the appendix).

We state the result in a more precise way. Given two spin network states $s, s' \in \mathcal{H}_{kin}$ defined on the graphs Γ_s and $\Gamma_{s'}$ respectively; let $\Gamma_{ss'}$ be the graph that satisfies the following properties:

1. $\Gamma_s, \Gamma_{s'} \subset \Gamma_{ss'}$,
2. The graph $\Gamma_{ss'}$ is the 1-skeleton of a cellular decomposition $K_{ss'}$ of Σ,
3. The 2-complex $K_{ss'}$ is *minimal* in the sense that it has the minimal number of 2-cells. We define the set of *irreducible loops* $\alpha^{ss'}$ as the set of oriented boundaries of the corresponding 2-cells in $K_{ss'}$.

With these definitions, the argument above shows that

$$
\begin{aligned}
< Ps, s' > \;&:=\; \lim_{\epsilon \to 0} \frac{1}{N_p^{\epsilon}!} \sum_{\sigma(\{i\})} \; < \prod_{p^{\sigma(i)}} \delta(U_{p^{\sigma(i)}}) s, \; s' > \\
&= \; < \prod_{\gamma \in \alpha^{ss'}} \delta(U_{\gamma}) s, \; s' >,
\end{aligned} \tag{33}
$$

where U_γ denotes as usual the holonomy around an irreducible loop $\gamma \in \alpha^{ss'}$.

3.2.2. Diffeomorphism invariance

Using this we can explicitly see how the generalized projector P implements diffeomorphism invariance or more precisely homeomorphism invariance with our regularization. We can explicitly illustrate this with two simple examples which sketch the general idea which can easily extended to a proof.

First consider a spin network and its deformation by a diffeomorphism which acts trivially everywhere except for a region intersecting a single edge. The two spin networks are orthogonal in $\mathcal{H}_{kin}$. Their difference is represented on the left of

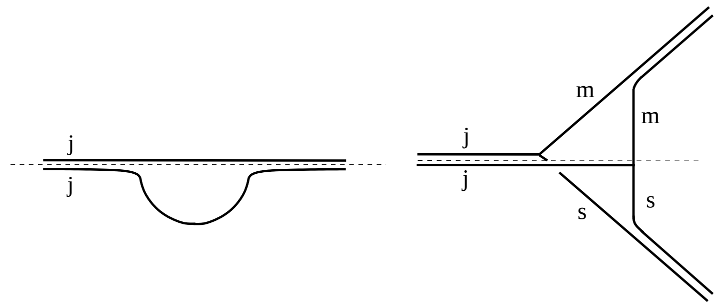
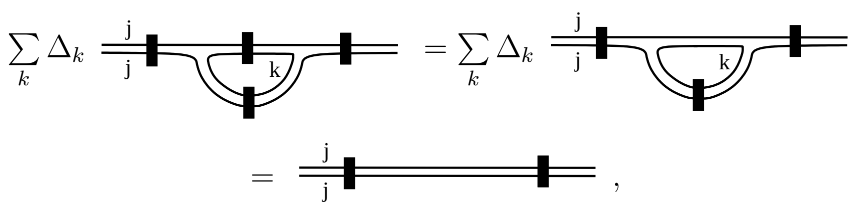


Figure 7: Examples of diffeomorphism equivalent (piece of) spin-networks. On the left, the diffeomorphism acts on the edge; on the right, it acts on the vertex.

Figure 7. Now, it follows from our definition of the physical inner product that the transition amplitude between the two spin network states is equal to one, namely

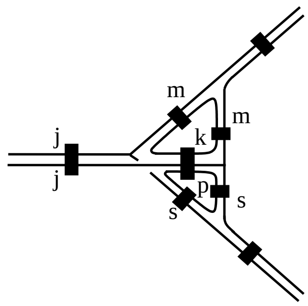


(34)

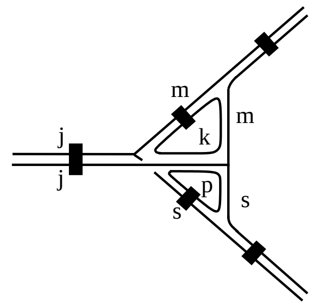
where in the first equality we have used the *gauge fixing* identity (47), and in the second we have used the *summation* identity (46). This shows that the deformed and undeformed states are physically equivalent and the equivalence is implemented by the operator P. More precisely, in the spin foam representation we will have a continuous transition between the initial spin network and the final one (obtained by the action of a diffeomorphism on the original one) with transition amplitude equals one.

In order to prove that any two spin networks belonging to the same homotopy class are physically equivalent we have to consider homeomorphisms that have non trivial action on spin network nodes. An example of this in the case of a 3-valent node is presented on the right of Figure 7. Again, the direct computation of the transition amplitude shows that the initial and final states are physically equivalent. Namely:

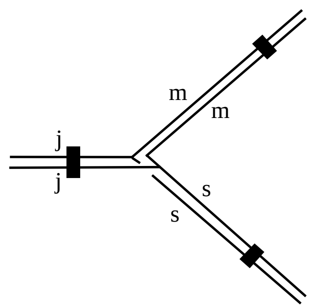
$$\sum_{k,p} \Delta_p \Delta_k$$



$$= \sum_{k,p} \Delta_p \Delta_k$$



$$=$$



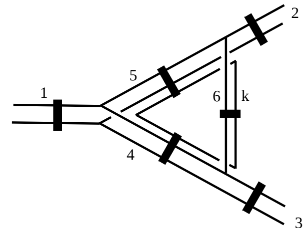
$$, \qquad (35)$$

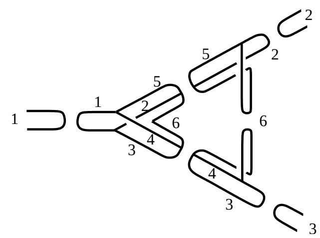
where again we have used the gauge fixing and summation identities. Combining the results of (34) and (35) we can prove that any two spin network states belonging to the same homotopy class are physically equivalent. At the classical level we had already seen that the curvature constraint (6) generates both diffeomorphism transformations in Σ in addition to time reparametrization that is represented by a fiducial notion of dynamics in the spin foam representation. We have seen how diffeomorphism invariance is imposed by the generalized projection P making elements of the homotopy class of spin networks physically equivalent.

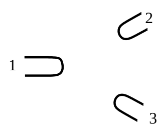
3.2.3. Relation with the Ponzano-Regge model

The rest of the equivalence is represented by the well known skein relations that relate physically equivalent spin network states in three dimensions which can be easily proved using our formalism. In three dimensions these relations imply that any spin network state is physically equivalent to some linear combination of spin network states based on certain minimal graphs which do not contain any irreducible loops. One of such skein relation follows directly from the transition amplitude corresponding to the process shown in Figure 5.

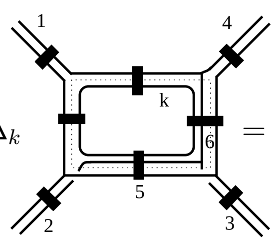
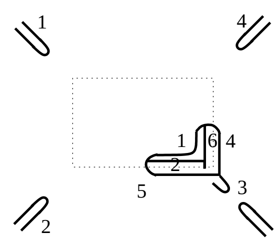
In the spin foam representation we have shown that such process can be represented by a sum over continuous spin foams as the one depicted in Figure 5. However, we have not yet computed its amplitude. Using the prescription of the irreducible loops the amplitude is simply given by the following:

$$\Delta_1\Delta_2\Delta_3\sqrt{\Delta_4\Delta_5\Delta_6}\sum_k \Delta_k$$ 

$$= \sqrt{\Delta_4\Delta_5\Delta_6}$$ 

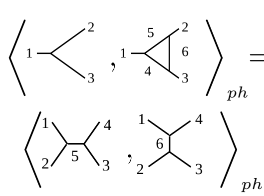
$$= \sqrt{\Delta_4\Delta_5\Delta_6}\left\{\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right\}$$  , (36)

where the tetrahedron represents the contraction of four 3-intertwiners which defines a $6j$-symbol and the factor $\Delta_1\Delta_2\Delta_3\sqrt{\Delta_4\Delta_5\Delta_6}$ comes from the normalization factors $\sqrt{\Delta_4\Delta_5\Delta_6}$ and $\sqrt{\Delta_1\Delta_2\Delta_3\Delta_4\Delta_5\Delta_6}$ in $\mathcal{H}_{kin}$ of the corresponding spin network states. The final amplitude on the right of the previous equation is clearly the Ponzano-Regge amplitude for that transition: in the spin foam picture the $6j$-symbol corresponds to the vertex amplitude and $\sqrt{\Delta_4\Delta_5\Delta_6}$ to the amplitude of the three new faces. In a similar way one can prove the re-coupling identity

$$\Delta_1\Delta_2\Delta_3\Delta_4\sqrt{\Delta_5\Delta_6}\sum_k \Delta_k$$  $= \sqrt{\Delta_5\Delta_6}$  , (37)

where amplitude on the right is again a $6j$-symbol.

We can write Equations (36) and (37) as



$$= \sqrt{\Delta_4\Delta_5\Delta_6}\left\{\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right\} \times \text{rest}$$

$$= \sqrt{\Delta_5\Delta_6}\left\{\begin{array}{ccc} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right\} \times \text{rest}, \quad (38)$$

Were by '*rest*' we denote the amplitude that follows from the details of the rest of the states at hand. The previous transitions correspond to the two Pachner moves that relate any two dual of simplicial decompositions in two dimensions. Any 2-complex

dual to a triangulation of $M = \Sigma \times \mathbb{R}$ in three dimensions can be foliated by a series of graphs that are dual to a triangulation of Σ. Therefore, we can reconstruct the amplitude of any 2-complex dual to a three dimensional triangulation which results in the well known Ponzano-Regge amplitude [27, 28, 30].

In our case there is an infinite series of skein relations as our states are based on arbitrary graphs when the regulator is removed.

3.2.4. Construction physical Hilbert space

At this stage we can give an explicit definition of the physical Hilbert space for any space time of topology $M = \Sigma \times \mathbb{R}$ where Σ is an arbitrary Riemann surface of genus $g > 1$. For the moment we will exclude the case $\Sigma = T^2$ which will be analyzed later (the case of the sphere has been treated in Section 3.1.). The spin foam representation of the projection operator introduced in this section allows us to select a complete basis in $\mathcal{H}_{phys}$. The generalized projection operator P can be viewed as implementing the curvature constraint by group averaging along the orbits of the constraint on the elements of $\mathcal{H}_{kin}$. In the previous subsection we have explicitly shown how this implies the physical equivalence of spin networks in the same homotopy class.

In addition to deforming spin network states by homeomorphisms the action of P can also create or annihilate irreducible loops as shown in Figure 5. This process can however occur only in one of the two directions as previously emphasized: the spin foam representation of P contains only tree-like 2-complexes with no bubbles as a result of our definition. We need to find a minimal set of states labeling the equivalence classes of states under the action of the curvature constraint. In order to simplify this task we must go in the direction in which graphs are simplified by the elimination of irreducible loops. This can be obtained by a series of Pachner moves of the previous section that relate physical equivalent graphs. Let us show this in more detail.

Given a contractible region of an arbitrary graph we can eliminate all the irreducible loops contained in that region by a combination of the gauge fixing identity (47) and the summation identity (46). As an end result all the components of the generalized connection in that region must be set to $\mathbb{1}$. If we have n outgoing links (labeled by $j_1 \cdots j_n$) from that region, then the state is physically equivalent to some linear combination of the finite orthonormal basis of intertwiners in $\mathrm{Inv}[j_1 \otimes \cdots \otimes j_n]$. In the spin foam picture this equivalence is represented by an evolution from the original complicated state to the linear combination of intertwiners by the elimination of irreducible loops in a tree-like spin foam with no bubble. In this process we are simply moving along the gauge orbits generated by the curvature constraint. We can continue this process until we arrive to a set of irreducible spin network states which in the case of the $g = 2$ Riemann surface is shown in Figure 8. The generalization for higher genus is obvious. This set of irreducible spin network states is an over-complete family of states in $\mathcal{H}_{phys}$. By construction these states label the

equivalence classes of elements $< Ps| \in Cyl^*$. In the generic g case the elements of this family are labelled by $6g-3$ quantum numbers as it can be checked.

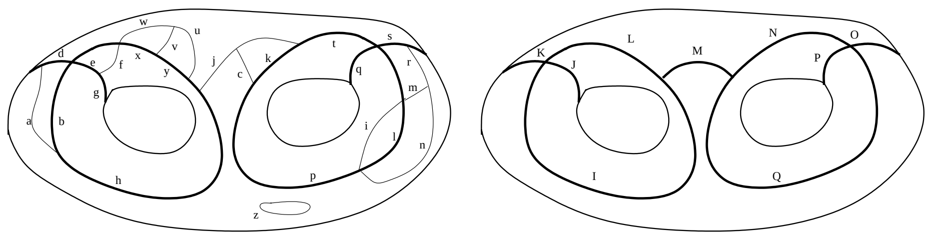


Figure 8: States on a genus two Riemann surface: Any arbitrary linear combination of spin network states on the left of Figure 8 labeled by quantum numbers a to z can be 'evolved' to a linear combination of spin network states on the right (labeled by only 9 spins). This is however still an over-complete generators' family of $\mathcal{H}_{phys}$.

Now we will propose a construction of a basis of $\mathcal{H}_{phys}$ based on the idea of gauge fixing the remaining gauge degrees of freedom in the previous over complete basis. Computing the inner product between the elements of this over complete basis involves a single irreducible loop. It is given by the contractible loop that can be drawn by contouring the graph on the right in Figure (8). The delta distribution associated to that last loop imposes what is left of the curvature constraint and is explicitly given by the following delta function:

$$\delta(g_1 g_2 g_1^{-1} g_2^{-1} g_3 g_4 g_3^{-1} g_4^{-1} \cdots g_{2g-1} g_{2g} g_{2g-1}^{-1} g_{2g}^{-1}) = \int dN \exp(i\mathrm{Tr}[NU_\ell^g]), \qquad (39)$$

where g_i are the holonomies around the $\Pi^1(\Sigma)$ generators and U_ℓ^g denotes the argument of the δ-distribution. We have re-expressed the δ as the integral of the exponentiated constraint to present the argument that follows. Formally the operator $\exp(i\mathrm{Tr}[NU_\ell^g])$ produces gauge transformations whose orbits are 3-dimensional. This tell us that in our over-complete basis there are three too many quantum numbers. More precisely, the nine spins labeled $I, \cdots, Q$ correspond to eigenstates of the length operators associated to paths that are dual to the spin network graph. Clearly these operators are not Dirac observables as they do not commute with the last constraint.

The three superfluous quantum numbers can be regarded as the residual gauge degrees of freedom that remain after imposing all the local constraints up to the last global one of (39). They correspond to two remaining 'global' diffeomorphisms Σ and time reparametrization.

The physical Hilbert space can be characterized by the subset of spin network states obtained from the over-complete irreducible set considered above by setting one of the spin to zero[5] (e.g., $M = 0$ in Figure (8)). Notice that the new set of states

[5]Let us consider a simplified version of the situation at hand defined by a non relativistic particle

are labeled by precisely $6g-6$ quantum numbers corresponding to the expected number of physical degrees of freedom (in Figure (9) we illustrate the case $g=2$; the case $g>2$ is illustrated in Figure (10)). One could consider the fixing of the spin of the intermediate link to a value different from zero. That would correspond however to a partial fixation as only the magnitude of E is fixed. This can be seen in the fact that unphysical configuration remain represented by the finite set of possible spin labels for the adjacent links corresponding to the quantized directions for the operator E.

Notice that the number of quantum numbers is $6g-6$ corresponding to the dimension of the moduli space of $SU(2)$ flat connections on a Riemann surface of genus g. In this way we arrive at a fully combinatorial definition of the standard $\mathcal{H}_{phys}$ by reducing the infinite degrees of freedom of the kinematical phase space to finitely many by the action of the generalized projection operator P. The result coincides with the one obtained by other methods that explicitly use the fact that the reduced phase space is finite-dimensional.

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confined to a sphere with the constraint $\hat{\phi}=0$. The auxiliary Hilbert space is taken to be the space of continuous functions on the sphere. Namely, a general state can be represented by a wave function

$$\Psi(\theta,\phi)=\sum_{\ell,m} c^{\ell m} Y_{\ell m}(\theta,\phi)$$

where $Y_{\ell m}(\theta,\phi)=P^m_\ell(cos(\theta))e^{im\phi}$ are spherical harmonics expressed in terms of the Lagrange polynomials P^m_ℓ. The physical scalar product

$$<\Psi,\Phi>_{phys}=<\Psi,\sum_n e^{in\hat{\phi}}\Phi>=\int d\Omega\delta(\phi)\overline{\Psi(\theta,\phi)}\Phi(\theta,\phi)=\int d[\cos(\theta)]\overline{\Psi(\theta,0)}\Phi(\theta,0).$$

In terms of the elements of the $Y_{\ell m}$ basis of $\mathcal{H}_{kin}$—the analog of the spin network states in gravity—the physical inner product takes the form

$$<\ \ell\ m,\ \ell'\ m'>_{phys}=\sqrt{\frac{(2\ell+1)(2\ell'+1)}{16\pi^2}\frac{(\ell-m)!(\ell'-m')!}{(\ell+m)!(\ell'+m')!}}\ \int_{-1}^{1} dx\ P^m_\ell(x)P^{m'}_{\ell'}(x),$$

where we have made the substitution $x=\cos(\theta)$ and we have introduced the standard ket notation $|\ell,m>$ to denote the quantum states. As in the gravity case the kinematical orthogonal states are no longer orthogonal as it can be easily checked using the previous equation. The states $|\ \ell\ m>$ are an over complete basis of $\mathcal{H}_{phys}$. A complete basis can be obtained if we fix the value of the $\hat{L}_z$ operator (conjugate to the constraint) to the value 0, i.e. $m=0$. Moreover, for any (m,ℓ), we can always expand P^m_ℓ in terms of P^0_ℓ as follows:

$$P^m_\ell(x)=\sum_{\ell'}\alpha^{\ell'}_{\ell,m}\ P^0_{\ell'}(x).$$

Finally, this implies that

$$|\ \ell\ m>\underset{phys}{=}\ \sum_{\ell'}\alpha^{\ell'}_{\ell,m}\ |\ \ell'\ 0>$$

so that the $\{|\ \ell\ 0>\}$ can be taken as a basis of $\mathcal{H}_{phys}$.

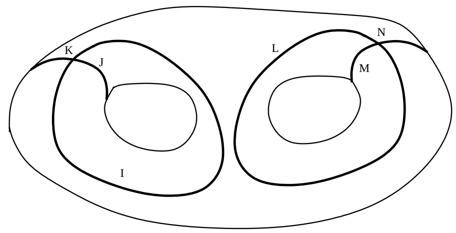

Figure 9: Example of a basis of $\mathcal{H}_{phys}$ for the genus $g = 2$ surface.

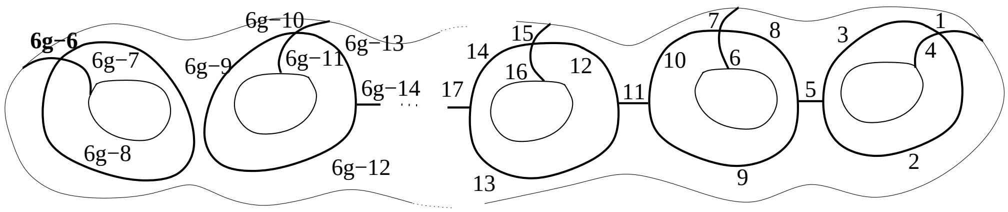

Figure 10: A basis of physical states for an arbitrary genus g Riemann surface. Our definition of the generalized projection P directly leads to such basis of physical states.

We must point out that one has still to show that the elements of the previous family are independent. We however think from the gauge fixing argument given above that this is in fact the case.

The case of the Torus ($g = 1$) is singular. It is however easy to show that the basis of the physical Hilbert space is generated, as a vector space, by the two loops as shown in Figure (11). The physical inner product defined in (20) is divergent and needs regularization. This problem is well known and has been investigated in the literature using reduced phase space quantization[39].

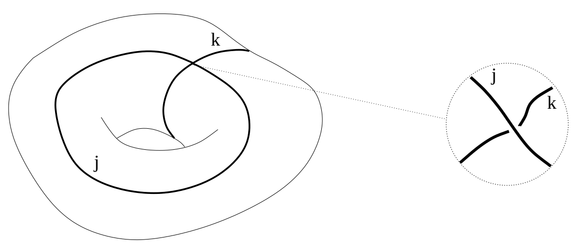

Figure 11: A basis of physical states in the case of the torus ($g = 1$).

4. Conclusion and discussion

Our paper establishes a clearcut connection between the canonical formulation of loop quantum gravity in three dimensions and previous covariant path integral definitions of the quantum theory. The Ponzano-Regge model amplitudes are recovered from the Hamiltonian theory and its '*continuum limit*' in the sense of Zapata [29] is

built in from the starting point. Divergences that plague the standard definition of spin foam models are not present and the formalism provides a clear understanding of their origin which is complementary to the covariant analysis provided in [26]. It also provides an explicit realization of Rovelli's proposal for resolving dynamics in loop quantum gravity.

We would like to emphasize the fact that we are getting fully background independent description of the spin foam in this case. The generalized projection operator P, providing spin foam amplitudes, is well defined independently of any background structure (such as a space time triangulation) and directly from the Hamiltonian picture. Our result is fully consistent with the standard formulation based on the quantization of the reduced phase space.

The simplicity of the reduce phase space quantization in 2+1 gravity might make reluctant to adopt the view point explored here. However, the reduced phase space quantization does not seem viable in 3+1 gravity. The spin foam perspective, fully realized here in 2+1 gravity, can bring new breath to the problem of dynamics in 3+1 gravity. Needless to say that the challenges in going to 3+1 are many and certainly no straightforward generalization of this work should be expected. In reference [31] an application of similar techniques is used to investigate a 'flat' solution to the constraints of Riemannian 3+1 gravity.

Generalization to BF theory in any dimensions seems straightforward. The new feature in higher dimensions is the fact that the Bianchi identity plays an important role in the regularization. If the dimension of Σ is larger or equal to 3 then the curvature constraints $F = 0$ are no longer independent since we also have that $dF = 0$. Through the non-Abelian Stokes theorem this implies that certain order integration of F on 2-dimensional closed surfaces vanishes. The regularization of P, introduced in Equation (26), must then be modified in order to avoid the inclusion of redundant delta functions.

The generalization to non vanishing cosmological constant is the natural next step. In this case the curvature constraint becomes $F^i - \Lambda \epsilon^i{}_{jk}\ e^j \wedge e^k = 0$. If $\Lambda > 0$ the theory can be quantized using the Chern-Simons formulation [32, 33] and the result involves the utilization of quantum groups. The 'path integral' version of the quantum theory is thought to correspond to the Turaev-Viro invariant of 3-manifolds [34, 35] which can be viewed as a generalization of the Ponzano-Regge model based on $U_q(su(2))$ for $q = \exp(i\frac{2\pi}{k+2})$ where $k = \Lambda^{-1/2}$. It would be important to understand whether these results can be obtained from the loop quantum gravity perspective presented here. The question is whether there is a well defined regularization of the projection operator P in this case and whether its definition implies the mathematical structures obtained by other means. Preliminary results indicate that this is the case. Namely, that one can start with the kinematical Hilbert space $\mathcal{H}_{kin}$ of (classical) $SU(2)$ spin networks and that value of the spin foam amplitudes derived from the arising from regularization of P in this case can be expressed in terms of the Turaev-Viro invariants and it generalizations [36].

In a companion paper [37] we use the formalism presented here to provide the complete quantization of general relativity coupled to point particles in three dimensions.

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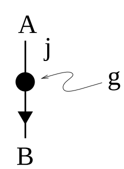
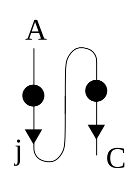
APPENDIX: $SU(2)$ Haar measure and useful identities

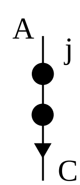
In this section we present the various identities that are used throughout the paper. The main identities where use in the case of $SU(2)$ in [36]. For a general treatment see [38]. We start by recalling the graphical notation that is commonly used in dealing with group representation theory. A spin j unitary irreducible $SU(2)$ representation matrix $\Pi^j(g)^A_B$ is represented by an oriented line going from the free index B and ending at A. The group element $g \in SU(2)$ on which the representation matrix is evaluated is depicted as a dark dot in the middle. Namely,

$$\Pi^j(g)^A_B =$$


. (40)

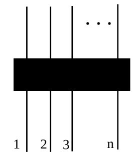
Matrix multiplication is represented by the appropriate joining of the line endpoints, more precisely

$$\Pi^j(g)^A_B \Pi^j(h)^B_C =$$


$=$


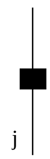
. (41)

For simplicity, we will drop the explicit reference to the free indices at the tips of the representation lines. The tensor product of representation matrices is simply represented by a set of parallel lines carrying the corresponding representation labels and orientation. An important object is the integral of the tensor product of unitary irreducible representations. We denote the Haar measure integration by a dark box overlapping the different representation lines as follows:

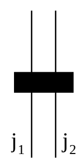
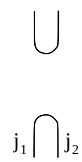
$$I^{A_1\cdots A_n}_{B_1\cdots B_n} = \int dg \Pi^1(g)^{A_1}_{B_1}\Pi^2(g)^{A_2}_{B_2}\cdots\Pi^n(g)^{A_n}_{B_n} =$$  .

(42)

The invariance of the Haar measure implies $I^2 = I$ and the invariance of I under right and left action of the group; therefore, I defines the projection operator $I^{12\cdots n}: 1\otimes 2\otimes\cdots\otimes n \to \mathrm{Inv}[1\otimes 2\otimes\cdots\otimes n]$. With this in mind it is easy to write the basic identities that follow from the properties of the Haar measure, namely

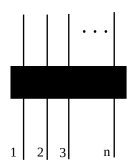
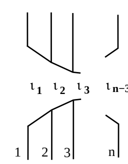
 $= \delta_{j,0},$

(43)

 $= \frac{1}{2j_1+1}\ \delta_{j_1 j_2}$  ,

(44)

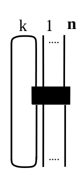
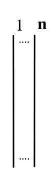
and generally

 $= \sum_{\iota_1\cdots\iota_{n-3}}$ 

(45)

where the right hand side is an expansion of the projector operator $I^{12\cdots n}: 1\otimes 2\otimes\cdots\otimes n \to \mathrm{Inv}[1\otimes 2\otimes\cdots\otimes n]$ in terms of a normalized basis of invariant vectors in $\mathrm{Inv}[1\otimes 2\otimes\cdots\otimes n]$. In the case of $SU(2)$ the invariant vectors in such basis can be labeled by $n-3$ half integers $\iota_1\cdots\iota_{n-3}$.

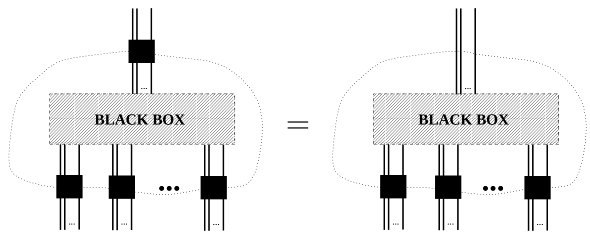
Other identities that we extensively use in this work where proved (at a more general level) in [36]. The first is the so-called *summation identity*

$\sum_k \Delta_k$  $=$ 

(46)

The other important identity proved in [36] is the so-called *gauge fixing identity*. Given a general graph with Haar measure integrations (represented by dark boxes)

if we can draw a close loop with no self-intersections that intersects the graph only through dark boxes, then we can erase one of the boxes without changing the evaluation. Graphically,



(47)

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